LINEAR MATROID INTERSECTION IS IN QUASI-NC

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[STOC-2018]

Definations

A Matroid is a pair $(E, \mathcal{I})$

where $E = [m]$ for some $m \in \mathbb{Z}$
(ground set)

$\mathcal{I} \subseteq P(E)$
(Independent sets)

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- $\forall\ I, J \in \mathcal{I}$ with $|I| < |J|$
  $\exists\ s \in J - I$ s.t. $I \cup \{s\} \in \mathcal{I}$
  (Augmentation)

Definations

For a Matroid $M := (E, \mathcal{I})$

- For $S \subseteq E$, rank of S is the maximal independent subset of S

  (rank function is submodular)

- Maximum Ind. sets in $\mathcal{I}$ are called Bases of M

- $\mathcal{B}$ will be set of all bases of M

Definations

M is a linear Matroid if

$\exists$ a matrix G_M s.t. $\forall I \in \mathcal{I}$

The rows in G_M index I are L.I.

Problem Statement

Given $M_1 := (E, \mathcal{I}_1)$ & $M_2 := (E, \mathcal{I}_2)$ two linear matroids, decide wheather

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$$

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$$\mathcal{I}_1 \cap \mathcal{I}_2 = \phi$$

$$\wedge\!\!\backslash_{NC}$$

Given $M_1 := (E, \mathcal{I}_1)$ & $M_2 := (E, \mathcal{I}_2)$ two linear matroids of rank n

decide wheather $\mathcal{I}_1 \cap \mathcal{I}_2 = \phi$

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Given $M_1 := (E, \mathcal{I}_1)$ & $M_2 := (E, \mathcal{I}_2)$ two linear matroids, decide wheather

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$$\bigwedge\!\!\backslash\!\backslash_{NC}$$

Given $M_1 := (E, \mathcal{I}_1)$ & $M_2 := (E, \mathcal{I}_2)$ two linear matroids of rank n

decide wheather $\mathcal{B}_1 \cap \mathcal{B}_2 \overset{?}{=} \phi$

Motivation

Edmonds' Problem

Given a Matrix polynomial

$$A(x_1, \dots, x_n) := x_1 A_1 + \dots + x_n A_n$$

where $A_i \in M_d(\mathbb{C})$

decide whether $\det(A(\bar{x})) = 0$

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decide whether $\det(A(\bar{x})) = 0$

— Solving this problem black-box will give Super-polynomial Lower Bound

(Hence hard to prove 🙂)

Motivation

Restricted Cases

- When A_is are rank 1 symbolic matrices
  ≡ Solving Bipartite Perfect Matching
  (Quasi-NC, [Fenner–Gurjar–Thierauf '16])
- When A_is are rank 1 matrices
  ≡ Solving Linear Matroid Intersection
  (Quasi-NC, [Gurjar–Thierauf '18])
- When A_is are rank-2 Skew-Sym. matrices
  ≡ Solving Linear Matroid Matching
  (white-box polytime [Lovász et. al])

- White box - polytime when X_is are Non-com.
- Deterministic aprx. algo for general 'search' problem.

Reduction

(Edm $\leq$ LMI)

Given $\sum A_i x_i$

Let $A_i = a_i \otimes b_i$

Then take $M_1 = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}_{d\times n}$

$M_2^T = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}_{d\times n}$

Find the intersection btn the matroids for M_1 & M_2

Reduction

Edm $\geq$ LMI)

Given M_1 & M_2 Linear Matroids of same rank (say n)

Let A_i be the matrix corresponds to M_i

Find det. of $\sum_{i=1}^{n} x_i (A_1)_i \otimes (A_2)_i$

$((A_i)_j \to j^{th}$ row of $A_i)$

Reduction

Proof:

$$\mathrm{Det}\left(\sum_{i=1}^{n} x_i (A_1)_i \otimes (A_2)_i\right)$$

$$= \mathrm{Det}\left(A_1\, I^{x}_{n\times n}\, A_2^{T}\right)$$

[$I^{x}_{n\times n}$ → diag. mat. with entries x_i at the diag.]

$$= \sum_{\substack{B \subseteq E \\ |B| = n}} \bar{x}^{B} \underbrace{[A_1]_B [A_2]_B^{T}}$$

$\boxed{\text{Non-zero when } B \text{ is a Common-Base}}$

Reduction

$(BPM \leq LMI)$

Given $G := (V_R \cup V_L, E)$ Bipartite graph

$(|E| = m)$

Define the matroid

$$M_L := (E, \mathcal{I}_L)$$

where $E \supseteq I \subseteq \mathcal{I}_L \iff |I \cap B_v| \leq 1$
$\forall v \in V_L$

$(B_v \subseteq E$, Set of edges incident at $v)$

Similarly define M_R

Find $M_L \cap M_R$

Reduction

Other problems can be solved from LMI :-

- Matroid Union
- Max. Rank Matrix Completion
- Rain-bow spanning spanning tree in edge-coloured graph
- Shortest R-S biconnector and a longest R-S biforest of a graph

⋮

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$$= \mathrm{Det}\left(A_1 I^x_{n\times n} A_2^T\right)$$

$$= \sum_{\substack{B \subseteq E \\ |B|=n}} \bar{x}^B [A_1]_B [A_2]_B^T$$

This gives an RNC^2 Algorithm for LMI

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This gives an RNC^2 Algorithm for LMI

- Apply random small weights to the exponent of x_i

  w.h.p. preserves non-zeroness

- Det. computation is in NC^2

Reduction

Proof:

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This gives an RNC^2 Algorithm for LMI

Here weight-assignments isolate the bases

Reduction

We will give W a set of weight-assignments. with the promise one of them is Isolating

$$|W| = 2^{\log^2 m}$$

$$\forall\, w \in W\,, \quad w = 2^{O(\log^2 m)}$$

Reduction

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$$\forall\, w \in W \,,\quad w = 2^{O(\log^2 m)}$$

Hence Quasi-NC Algorithm

More Definitions :)

Matroid Polytope:

for $S \subseteq E$, define

$x^S \in \{0,1\}^{|E|}$ The characteristic vector of S

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Matroid Polytope:

for $S \subseteq E$, define

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For a $\mathcal{G}$:= family of subsets of E

Define $P(\mathcal{G})$

by the convex hull of

$$\{x^I \mid I \in \mathcal{G}\}$$

Define the Matroid Polytope

by $P(\mathcal{I})$

Matroid Polytope

Edmonds' Characterisation

[Edmonds '70]

For $x \in \mathbb{R}^E$

$$x \in P(\mathcal{I}) \iff \begin{array}{l} x_e \geq 0 \quad \forall e \in E \\ x(S) \leq r(S) \\ \quad \forall S \subseteq E \end{array}$$

Matroid Polytope

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$x \in P(\mathcal{B}) \iff x_e \geq 0 \ \forall e \in E$

$x(S) \leq r(S)$

$\forall S \subseteq E$

$x(E) = n$

For the rest of out talk

$$\left.\begin{array}{l} M_1 := (E := [m], \mathcal{I}_1) \\ M_2 := (E, \mathcal{I}_2) \end{array}\right\} \text{Inputs}$$

$\mathcal{B}_i \longrightarrow$ Set of bases

Matroid Polytope

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For $x \in \mathbb{R}^E$

$$x \in P(\mathcal{I}_1 \cap \mathcal{I}_2) \iff \begin{array}{l} x_e \geq 0 \quad \forall e \in E \\ x(S) \leq r_1(S) \\ x(S) \leq r_2(S) \\ \quad \forall S \subseteq E \end{array}$$

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Matroid Polytope

Edmonds' Characterisation

$$P(\mathcal{I}_1 \cap \mathcal{I}_2) \stackrel{?}{=} P(\mathcal{I}_1) \cap P(\mathcal{I}_2)$$

$$P(\mathcal{B}_1 \cap \mathcal{B}_2) = P(\mathcal{B}_1) \cap P(\mathcal{B}_2)$$

Goal: Find a singleton face in $P(\mathcal{B}_1 \cap \mathcal{B}_2)$

Matroid Polytope

Edmonds' Characterisation

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$$P(\mathcal{B}_1 \cap \mathcal{B}_2) = P(\mathcal{B}_1) \cap P(\mathcal{B}_2)$$

All the corner points $\subseteq \{0,1\}^{|E|}$

Matroid Polytope

Weight-assignment:

A weight-function $\omega: E \to \mathbb{Z}$ can be extended to polytopes

$$\omega: \mathbb{R}^E \to \mathbb{R}, \quad x \longrightarrow \omega \cdot x$$

Matroid Polytope

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A weight-function $\omega : E \to \mathbb{Z}$ can be extended to polytopes

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Let $T \subseteq P(\mathcal{B}_1 \cap \mathcal{B}_2)$ containing the points x where $\omega(x)$ is minimum

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$\boxed{\text{Claim: } T \text{ is a Face}}$

Matroid Polytope

Partition lemma (characterising Faces)

If F is a face of $\mathcal{P}(\mathcal{B})$

$\exists$ a partition $\mathcal{G}$ of E s.t.

- $\forall S \in \mathcal{G}$, $\exists\, n_S \in \mathbb{Z}_{\geq 0}$, s.t.

  $$x(S) = n_S$$

- $\forall T \subseteq E$ s.t. $x(T) = r(T)$ $\forall x \in F$

  T is disjoint union of elements from $\mathcal{G}$

- $\forall e \in E$ s.t. $x_e = 0$, $\forall x \in F$,

  $$\{e\} \in \mathcal{G}, \; n_{\{e\}} = 0$$

Matroid Polytope

Partition lemma (Characterising Faces)

If F is a face of $P(B_1 \cap B_2)$

$\exists$ a partition $\mathcal{G}_1, \mathcal{G}_2$ of E s.t.

- $\forall S \in \mathcal{G}_i$, $\exists\, n_s, m_s \in \mathbb{Z}_{\geq 0}$, s.t

  $$x(S) = n_s / m_s$$

- $\forall\, T \subseteq E$ s.t. $x(T) = r_i(T)$ $\forall x \in F$

  T is disjoint union of elements from $\mathcal{G}_i$

- $\forall\, e \in E$ s.t. $x_e = 0$, $\forall x \in F$,

  $\{e\} \in \mathcal{G}_1, \& \ \mathcal{G}_2$. $n_{\{e\}} = 0 = m_{\{e\}}$

Matroid Polytope

Let F be a face of $P(\mathcal{B}_1 \cap \mathcal{B}_2)$

$C := \{e_1, \dots, e_{2r}\} \subseteq E$ is called cycle

if $\forall\ i \in [r]$

$$e_{2i-1}, e_{2i} \in S_i \text{ for some } S_i \in \mathcal{G}_1$$
$$e_{2i}, e_{2i+1} \in T_i \text{ for some } T_i \in \mathcal{G}_2$$

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Obs: For B_1, B_2 bases in $P(\mathcal{B}_1 \cap \mathcal{B}_2)$

$B_1 \Delta B_2$ is set of disjoint Cycles

Matroid Polytope

Let F be a face of $P(\mathcal{B}_1 \cap \mathcal{B}_2)$

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Obs: For B_1, B_2 bases in $P(\mathcal{B}_1 \cap \mathcal{B}_2)$

$B_1 \Delta B_2$ is set of disjoint cycles

Cor: $C_F = \emptyset \Rightarrow F$ is a point

(C_F : Set of cycles for F)

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<u>Circulation on C</u> : For a wt-assignment w, define

$$c_w(C) := \left| w(e_1) - w(e_2) + \cdots - w(e_{2r}) \right|$$

Isolating wt-assignment

Lemma: F be a Face of the polytope $P(\mathcal{B}_1 \cap \mathcal{B}_2)$

Suppose for some wt-assignment ω

$\omega(x)$ is const. $\forall\, x \in F$

Then $c_\omega(c) = 0 \quad \forall\; c \in \mathcal{C}_F$

Isolating wt-assignment

Lemma: F be a Face of the polytope $P(B_1 \cap B_2)$

Suppose for some wt-assignment ω, $\omega(x)$ is const. $\forall x \in F$

Then $c_\omega(c) = 0 \quad \forall \; c \in C_F$

Cor: If ω ensures non-zero circulations for all cycles in $P(B_1 \cap B_2)$

ω isolates a corner point

Isolating wt-assignment

Lemma: We can construct $O(m^2 s)$ wt-functions, each wt bounded by $O(m^2 s)$, to ensure non-zero circulation for s cycles

[Fredman–Komlos–Szemerédi '84]

There are Exp. many Cycles 🥲

Isolating wt-assignment

Lemma: We can construct $O(m^2 s)$ wt-functions, each wt bounded by $O(m^2 s)$, to ensure non-zero circulation for s cycles

Cor: $O(m^6)$ wt-functions are needed, each bounded by $O(m^6)$ to ensure non-zero circulations for $\le m^4$ cycles
(i.e., All possible 4-length cycles)

Isolating wt-assignment

<u>lemma</u>: $F \to$ Face of $P(B_1 \cap B_2)$

if C_F has no cycle of length r (≥ 2), C_F will have $\leq m^4$ cycles of length $2r$

Isolating wt-assignment

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By this lemma we will construct

$W_0, \ldots, W_t$ round wt. function

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Now $N\omega_0 + \omega_t$ $(\omega_0 \in W_0, \omega_t \in W_t)$ will give W_{t+1}

(N is some number $\geq \omega \in W_0$
$\therefore N = m^7$ enough)

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$$\boxed{|W_t| = |W_0|^t = m^{6t}}$$

Isolating wt-assignment

By this lemma we will construct

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We will stop at $t = \lceil \log m \rceil$

as cycle-length can be atmost m

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$W_{\lceil \log m \rceil}$ is our output

Isolating wt-assignment

$W_{\lceil \log m \rceil}$ is our output

Each $w \in W_{\lceil \log m \rceil}$ is of the form

$$\sum_{i=1}^{\lceil \log m \rceil} N^i w_i \qquad (w_i \in W_0)$$

$$\therefore \quad w \leq N^{\log m} m^6 \leq (m^7)^{\log m}$$

and $\quad |W_{\lceil \log m \rceil}| = O(m^{6 \log m})$

THANK

YOU