Exponential Lower-bounds via Exponential Sums

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Outline

NP Problems: Is Brute Force Optimal?

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- **ETH** says NO! (informally)

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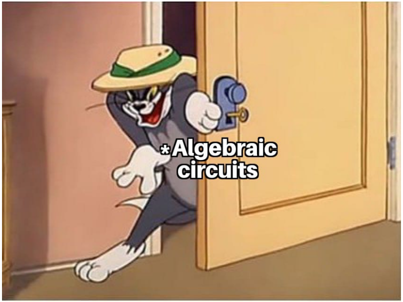
- Hence Algebraic Circuit enters the picture



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- Complexity Measure: $\#$ Edges in the circuit

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We believe these are $\geq \Omega(n)$

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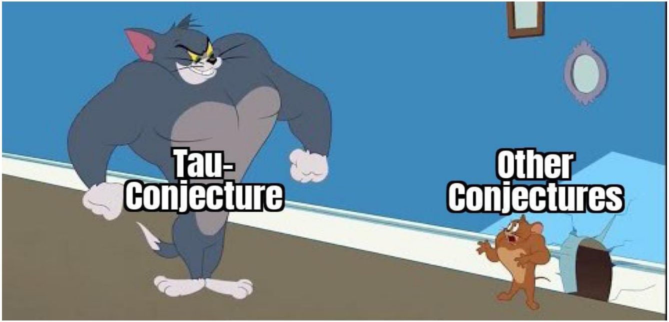

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- Intuitively polynomials with small size circuits

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2. $S_{n,k}(X_1, \ldots, X_n) := \sum_{S \subseteq [n], |S|=k} \prod_{i \in S} X_i$
3. $Det_n(\mathbf{X}) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{1,\sigma_1} X_{2,\sigma_2} \ldots X_{n,\sigma_n}$

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- Does τ-conjecture imply some lower bound?
- [Bürgisser'07] showed super-polynomial lowerbound on $P_{m,n}$ assuming τ-conjecture

Main result

Conditional Optimal Lower Bound [BBDM'24]

Assuming τ-conjecture $\exists$ a polynomial family $P_{n,m}(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}]$ of exponential sum which requires $2^{\Omega(n)} poly(m)$ size circuit.

Bürgisser's Proof analysis

Assume every $P_{n,m}(\mathbf{X})$ has $poly(m)$ circuit

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$$\prod_{i=1}^{n}(x+i) \text{ has easy coefficients}$$

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It has poly($\log n$) size circuit

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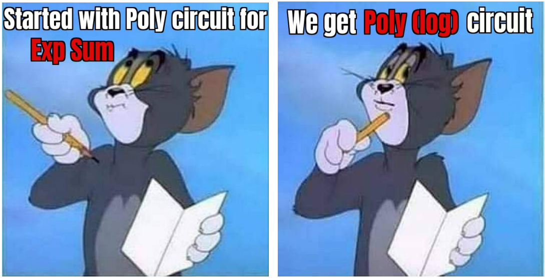

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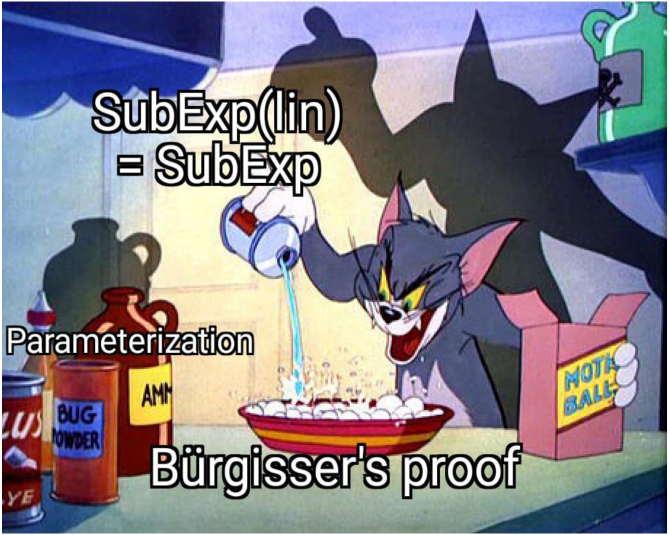
But we can improve it!



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where

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What's the Magic

$$S_{n,n}x^n + S_{n,n-1}x^{n-1} + \cdots + S_{n,0}$$ **3 level**

$$S_{n,k} := 1 \times \cdots \times k + 2 \times \cdots \times (k+1) + \ldots$$ **2 level**

$$1 \times \ldots \times k, \quad 2 \times \ldots \times (k+1), \quad \ldots$$ **1 level**

$$1, 2, \ldots, n, \quad x^k$$ **0 level**

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and some polynomial time computable function $f : \{0,1\}^* \to \mathbb{N}$
such that,

$$x \in A \iff |\{y \in \{0,1\}^{\ell(|x|)} : \langle x, y \rangle \in B\}| > f(x).$$

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The *linear counting hierarchy* is $\text{CH}_{\text{lin}} K := \bigcup_{k \geq 0} \text{C-lin}_k K$

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Hence Exponential sum is EASY $\implies$ CH_{lin} collapses

$\implies \prod_{i=1}^{n} (x+i)$ is EASY

Permanent

Given a variable matrix

$$\mathbf{X} := \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nn} \end{bmatrix}_{n \times n}$$

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$$Per_n(\mathbf{X}) := \sum_{\sigma \in S_n} X_{1,\sigma_1} X_{2,\sigma_2} \dots X_{n,\sigma_n}$$

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Other Results

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2. We achieved completeness result for parameterized valiant classes.

Open Problems

1. Can we established conditional truly exponential (ie, $2^{\Omega(n)} poly(n)$) lower bound for Per_n? (Unconditional will be better :))

2. Can we get Lower Bounds for *NP* from tau-conjecture? (We don't know even super-polynomial bound)