Deterministic factorization of constant depth algebraic circuits in subexponential time

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(submitted)

Outline

Disclaimer: We are *rational* person

Recall High-School Algebra

Factorize the polynomial:

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Fact: Univariate Factorization is Easy [LLL '82]
Multivariate not so sure

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- Efficient when polynomials are given via monomial representation

Algebraic circuit

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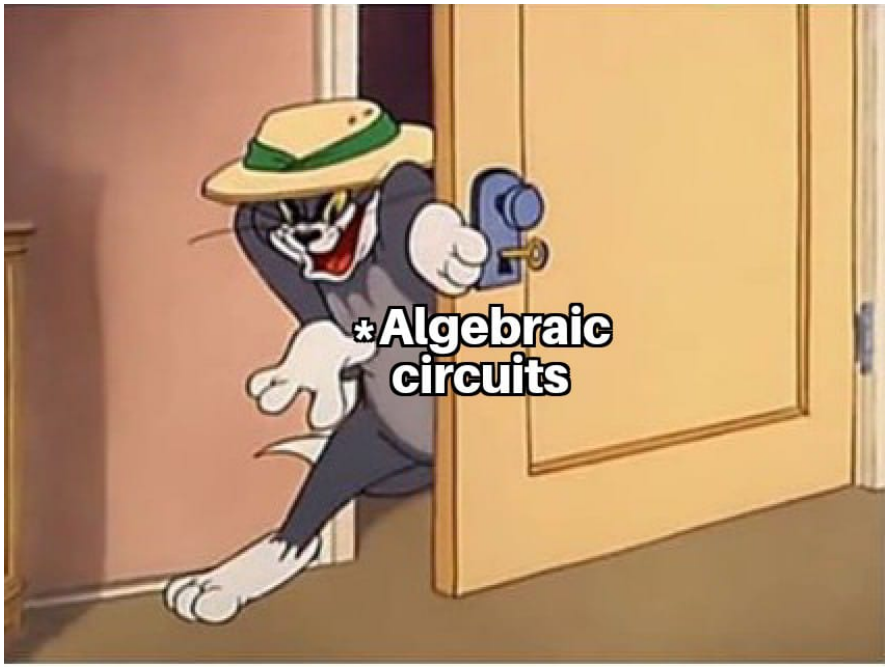
Hence Algebraic Circuit enters the picture



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- **Size**: # Edges in the circuit
- **Depth**: length of shortest paths from root to leaf

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- n many random bits needed

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Polynomial Factorization problem has applications in:

- Coding Theory
- Pseudorandomness
- Cryptography
- Graph algorithms

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- Can we factor constant depth circuits in sub-exp?
- [KRS 23], [DST24], [KRSV 24] made progresses towards this
- Finally we answered the question: Yes, we can!

Why so difficult

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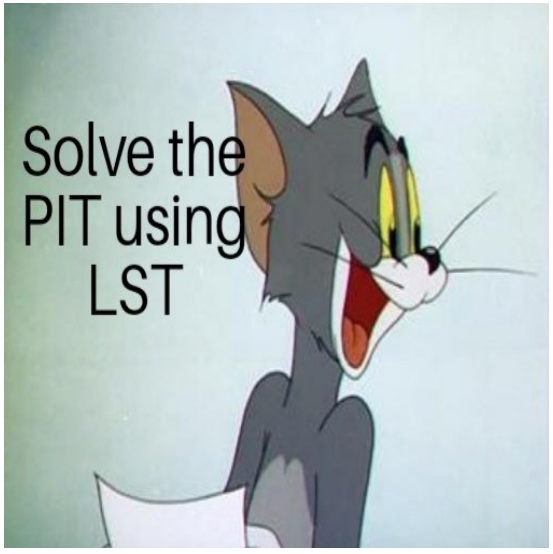

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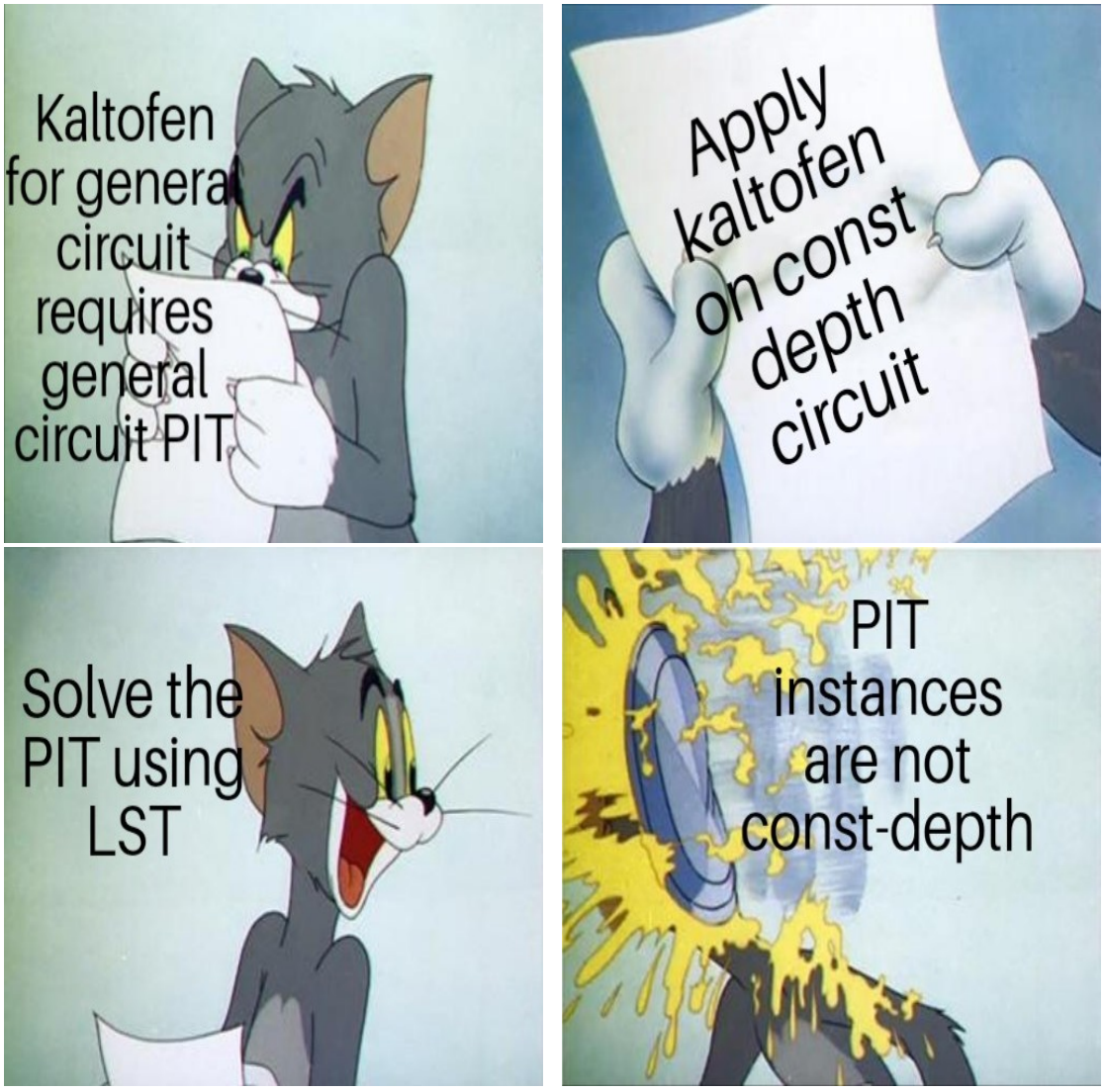

Newton Iteration/ Hensel Lifting destroyes the fun

Somehow we did it

Main Result

Theorem

Given constant $\varepsilon > 0$, *we can factor constant-depth circuit* $\mathcal{C}(x_1, \ldots, x_n)$ *of size* $poly(n)$ *in* $\tilde{O}(n^{n^{\varepsilon}})$

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Key Step: Solve the PIT instances in Kaltofen's Algorithm

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u is the constant in $\varphi(\mathbf{X})$

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- $\varphi^{(r)}$ has $O(r)$-depth circuit

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- That's where PIT comes in

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- Don't get confused :)

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- [LST 21]: Gave an explicit f
- [KI 04]: Gave an explicit variable reduction using such f which preserves non-zeroness.

Constant-depth-like PIT idea

Key lemma

Low degree roots of constant-depth like circuits also have small constant depth circuit

Highlevel idea: Newton iteration (with quadratic convergence)

Same [KI] variable reduction will work

Field dependency

- This algorithm is efficient in terms of bit-complexity
- Works for any field with $char > \omega(d)$ or $= 0$ and univariate factorization is known

Open questions

- Can we avoid the dependency on the hardness result? i.e., can we use the constant depth PIT result black-box to solve factorization?
- Are constant depth circuits closed under factorization?
- Can we do something better for sparse polynomials?